Polynomials Bloch Equations

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Polynomials Bloch Equation

- Polynomials Bloch equation

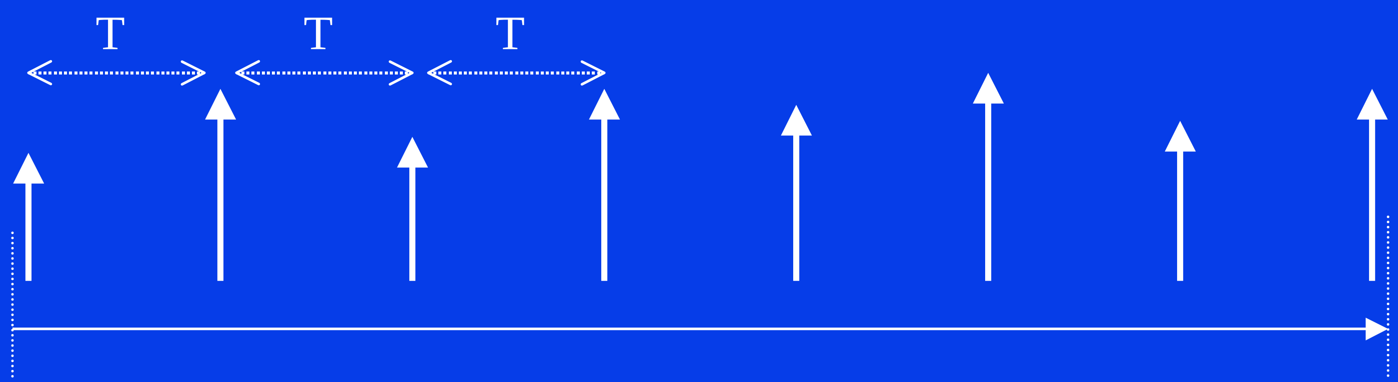
- Shinnar-Le Roux selective pulse design: polynomials Schroedinger equation.

Inverse Scattering Transform

M.H. Buonocore: MRM, 1993, V29, N4 (APR),P470-477

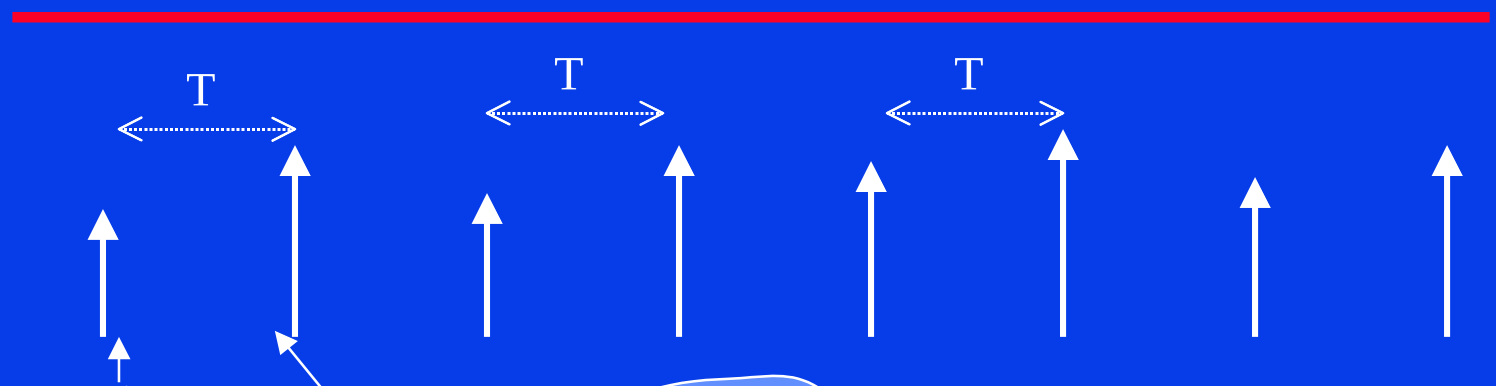
Polynomials Bloch Equation

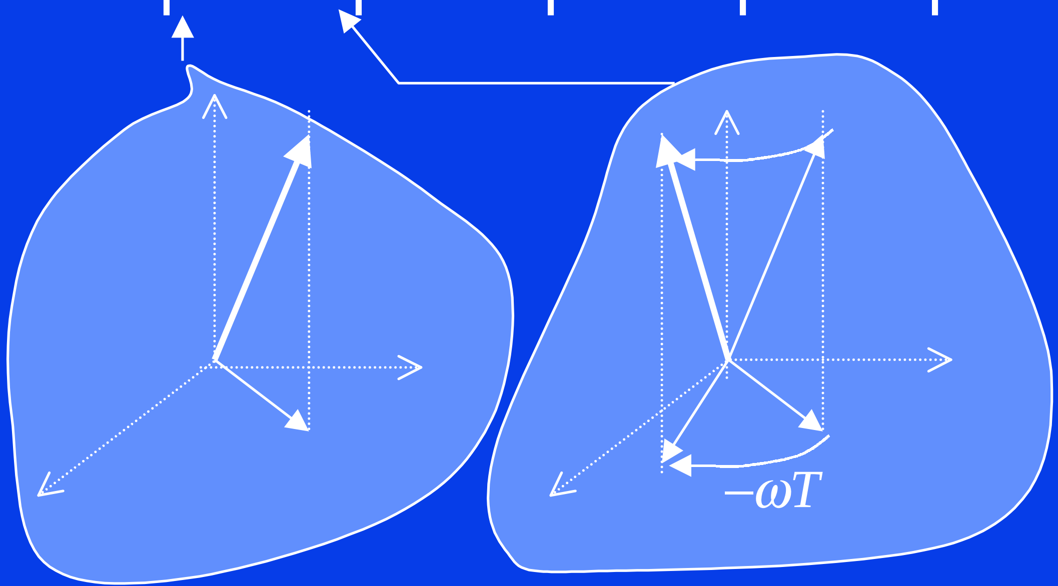
Train of hard pulses



Action of here to there

Free precession periods





$$M_x^+ = M_x \cos\omega T + M_y \sin\omega T$$

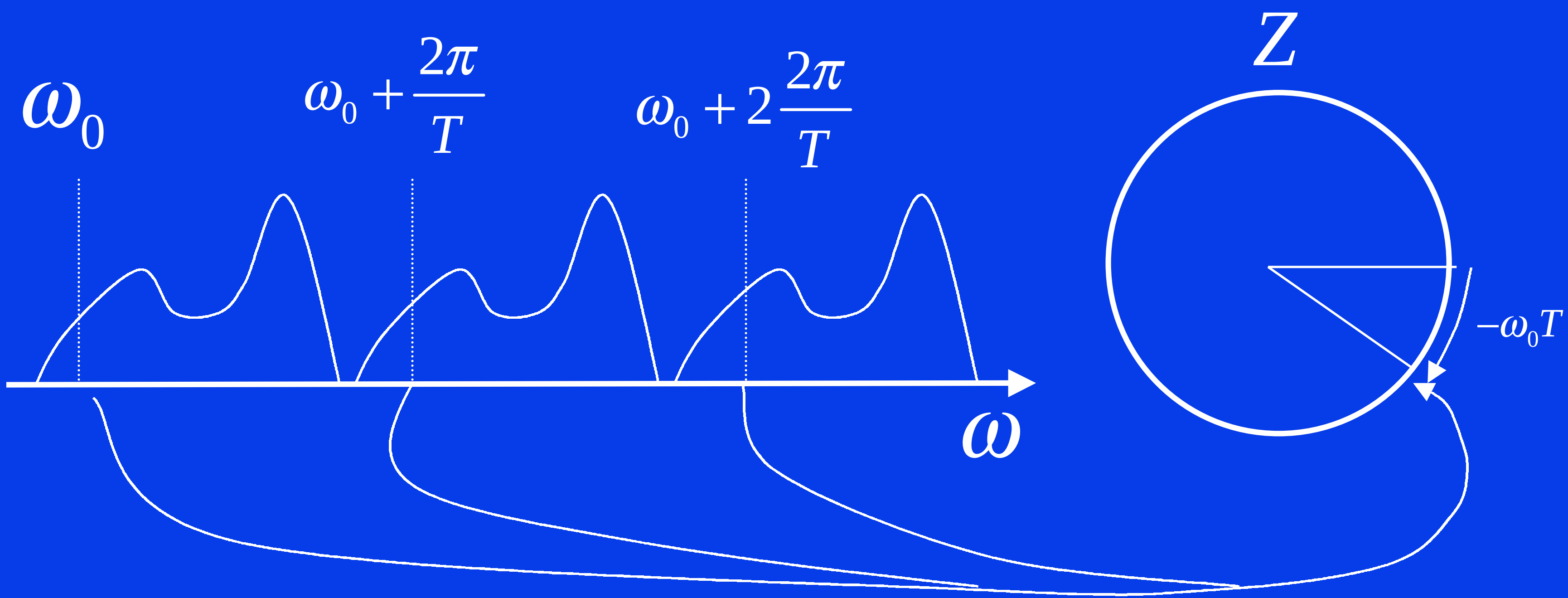
$$M_y^+ = -M_x \sin\omega T + M_y \cos\omega T$$

$$\boxed{M_{xy}^+ = e^{j(-\omega)T} M_{xy} = Z^{-1} M_{xy}}$$

$$\boxed{Z = e^{j\omega T}} \qquad M_{xy} = M_x + jM_y$$

Z plane, Z transform

$Z^{-1} = e^{-j\omega T}$ is the factor of phase advance of the proton (with resonance angular frequency ω) in a free precession period (it would be Z, for an electron).



Nutation (hard pulse)

Rotation around y, of angle θ

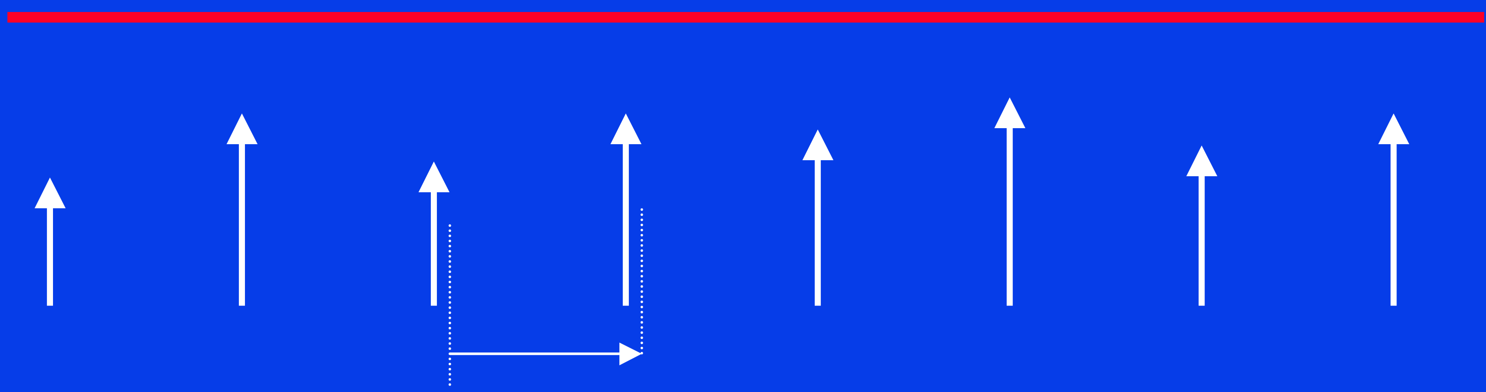
$$M_x^+ = \cos(\theta) M_x + \sin(\theta) M_z$$
$$M_y^+ = M_y$$
$$M_z^+ = -\sin(\theta) M_x + \cos(\theta) M_z$$

$$c = \cos(\frac{\theta}{2}) \quad s = \sin(\frac{\theta}{2})$$
$$\sin(\theta) = 2cs$$
$$\cos(\theta) = c^2 - s^2 = 2c^2 - 1 = 1 - 2s^2$$

$$M_x^+ = \cos(\theta) \frac{M_{xy} + M_{xy}^*}{2} + \sin(\theta) M_z$$
$$M_y^+ = M_y = \frac{M_{xy} - M_{xy}^*}{2}$$
$$M_z^+ = -\sin(\theta) M_x + \cos(\theta) M_z$$

$$M_{xy}^+ = 2cs\, M_z + c^2\, M_{xy} - s^2\, M_{xy}^*$$
$$M_z^+ = (c^2 - s^2) M_z - cs\, M_{xy} - cs\, M_{xy}^*$$

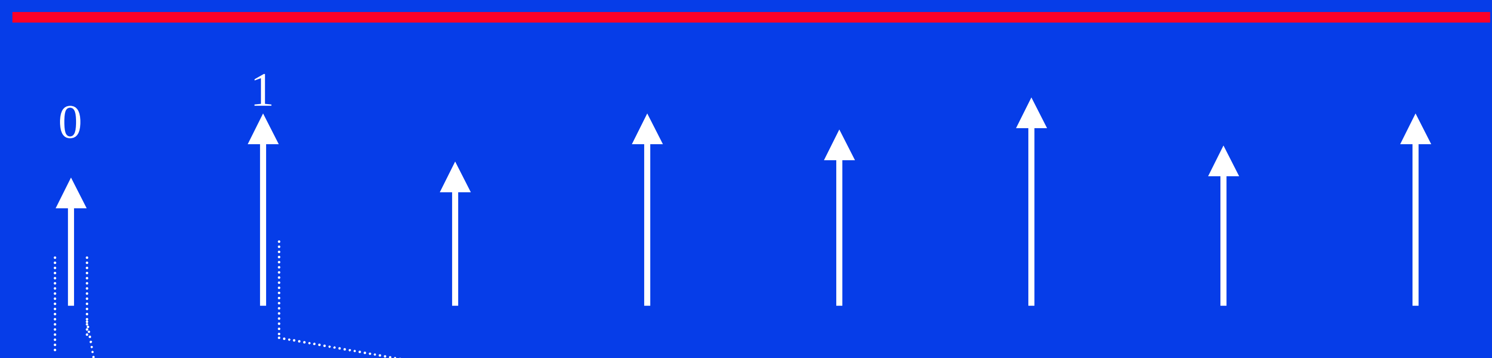
One period: precession and nutation



$$M_{xy}^{+} = 2cs\, M_z + c^2 Z^{-1} M_{xy} - s^2 Z\, M_{xy}^{*}$$

$$M_{z}^{+} = (c^2 - s^2) M_z - cs\, Z^{-1} M_{xy} - cs\, Z\, M_{xy}^{*}$$

With initial condition: M along Z



$$M_{xy} = 0$$

$$M_z = 1$$

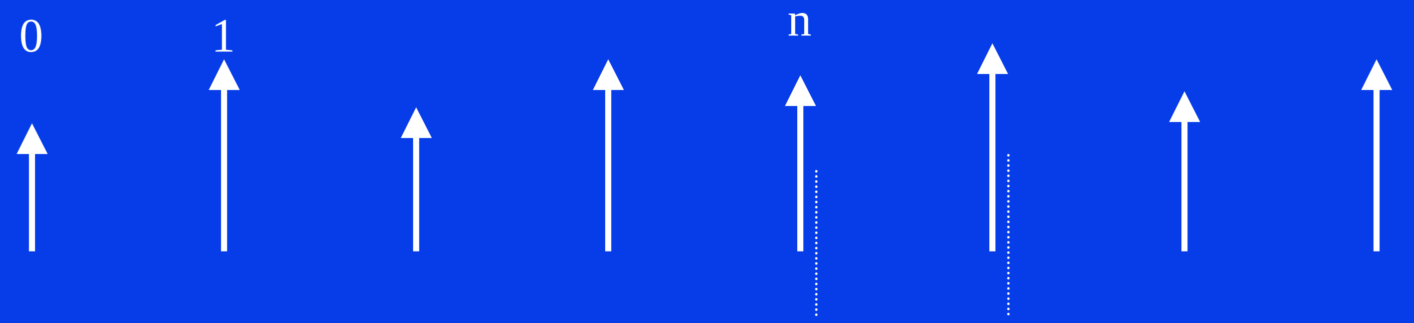
$$M_{xy} = 2c_1 s_1 (c_0^2 - s_0^2) + 2c_1^2 c_0 s_0 \boxed{Z^{-1}} - 2 s_1^2 c_0 s_0 \boxed{Z}$$

$$M_z = (c_1^2 - s_1^2)(c_0^2 - s_0^2) - 2c_1 s_1 c_0 s_0 \boxed{Z^{-1}} - 2c_1 s_1 c_0 s_0 \boxed{Z}$$

$$M_{xy} = 2c_0 s_0$$

$$M_z = (c_0^2 - s_0^2)$$

Recursion



$$M_{xy} = q_{-n}Z^{n} + \ldots q_{-1}Z + q_0 + q_1 Z^{-1} \ldots + q_n Z^{-n}$$

$$M_z = p_{-n}Z^{n} + \ldots p_{-1}Z + p_0 + p_1 Z^{-1} \ldots + p_n Z^{-n}$$

$$M_{xy} = q_{-n-1}Z^{n+1} + q_{-n}Z^{n} + \ldots q_{-1}Z + q_0 + q_1 Z^{-1} \ldots + q_n Z^{-n} + q_{n+1}Z^{-n-1}$$

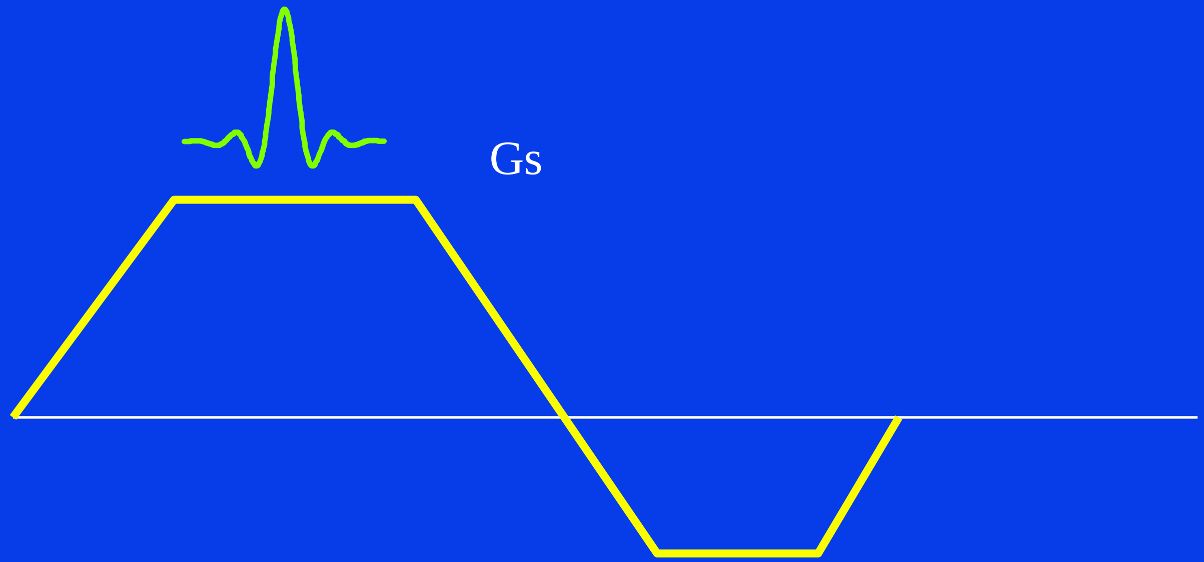
$$M_z = p_{-n-1}Z^{n+1} + p_{-n}Z^{n} + \ldots p_{-1}Z + p_0 + p_1 Z^{-1} \ldots + p_n Z^{-n} + p_{n+1}Z^{n+1}$$

Polynomials Bloch Equation

- Useful for Analysis, or simulation: I.e. for knowing the action of a given train of pulses.
- There is a Synthesis, ‘peel off’ algorithm permitting, in theory, to find the sequence of hard pulses yielding a prescribed response. But the preliminary specification of this response is very hard, and the algorithm itself is numerically unstable for large n, limiting the use to very short sequences. V.H. Subramanian, S.M. Eleff, S.Rehn, J.S. Leigh “An Exact Synthesis procedure for frequency selective pulses” 5th Proceeding of the Magnetic Resonance In Medecine SoC August 1986.
- Other use: FSE stabilization. P.LeRoux,S.Hinks,MRM 30,183-191,1993
- For soft, finite time pulses and without relaxation the SLR algorithm is the choice (Le Roux French Patent 86 10179 July 1986)

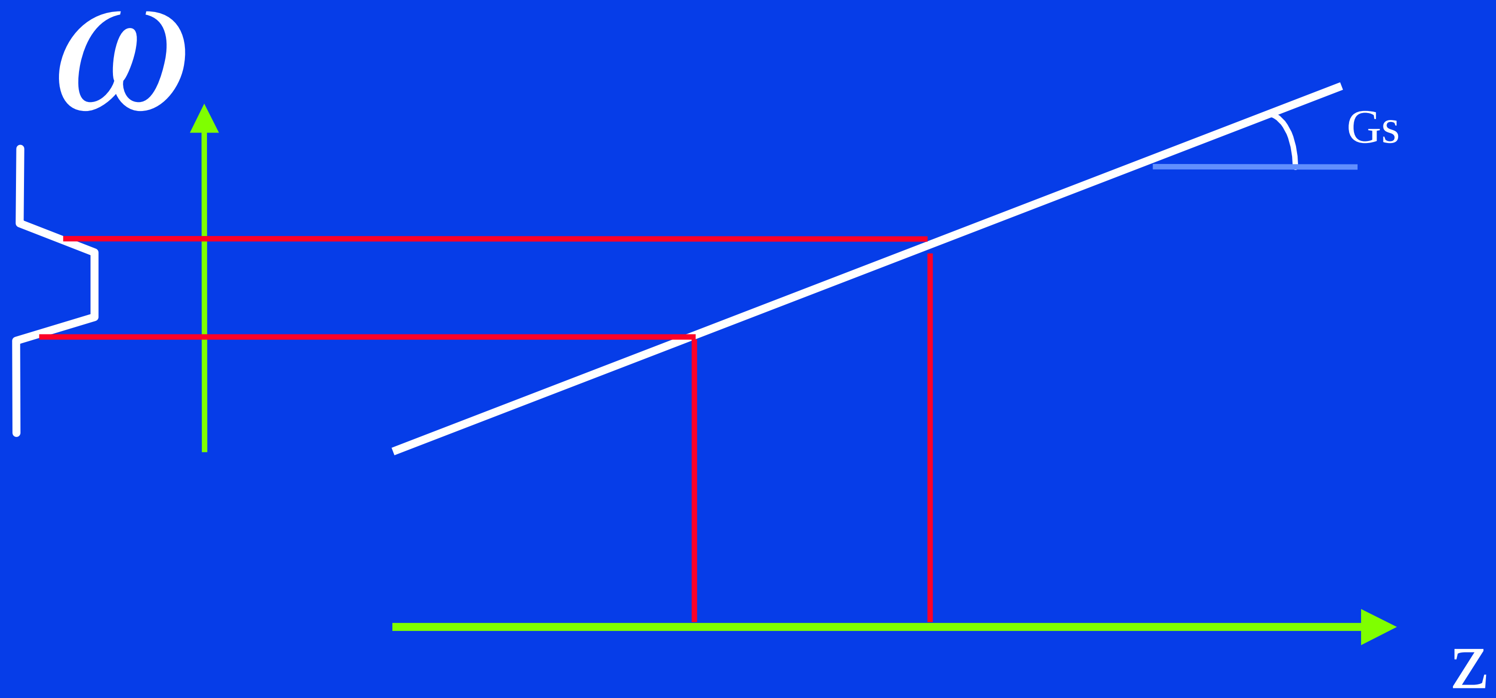
Shinnar-Le Roux:Slice Selection

- Principle



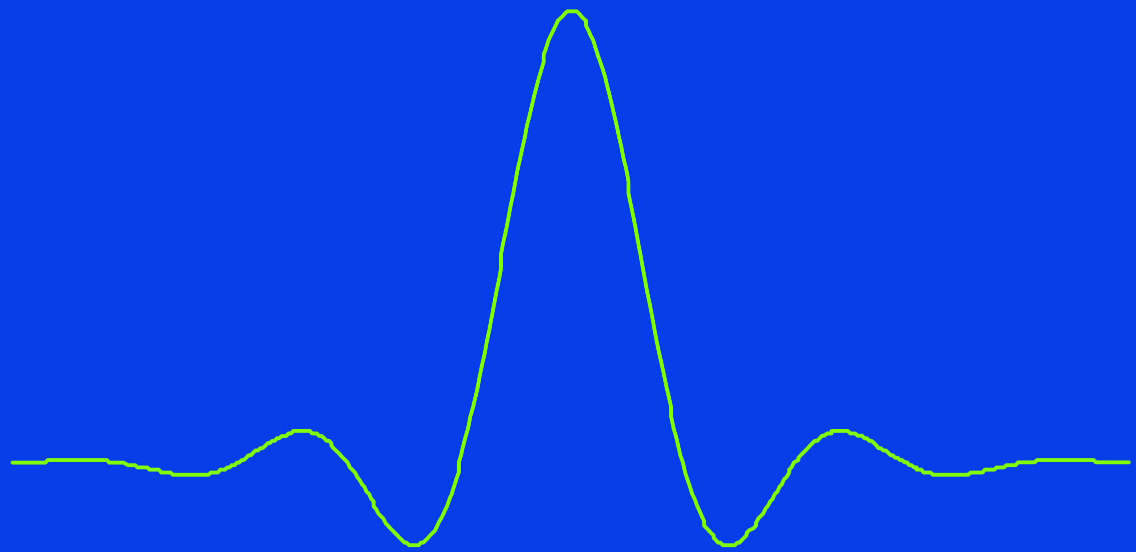
Slice Selection

- Principle



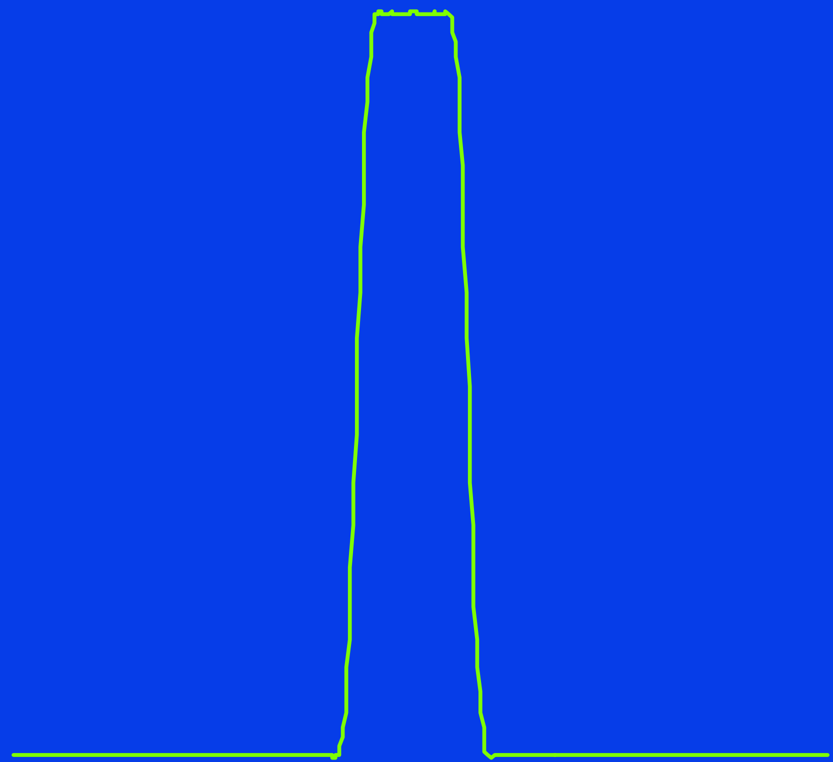
Selective (soft) Pulse

- A b1 waveform (apodized sinc)



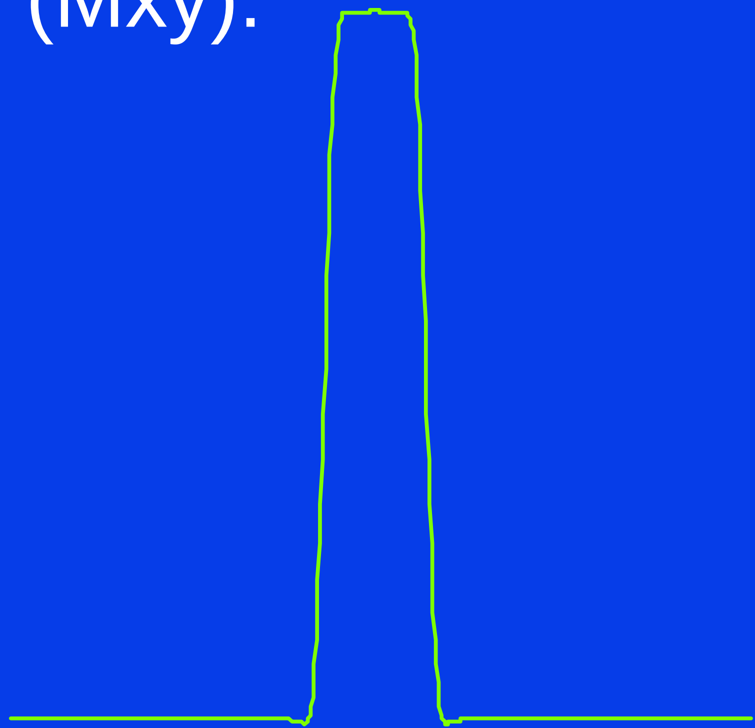
Selective (soft) Pulse

- Its Fourier transform:



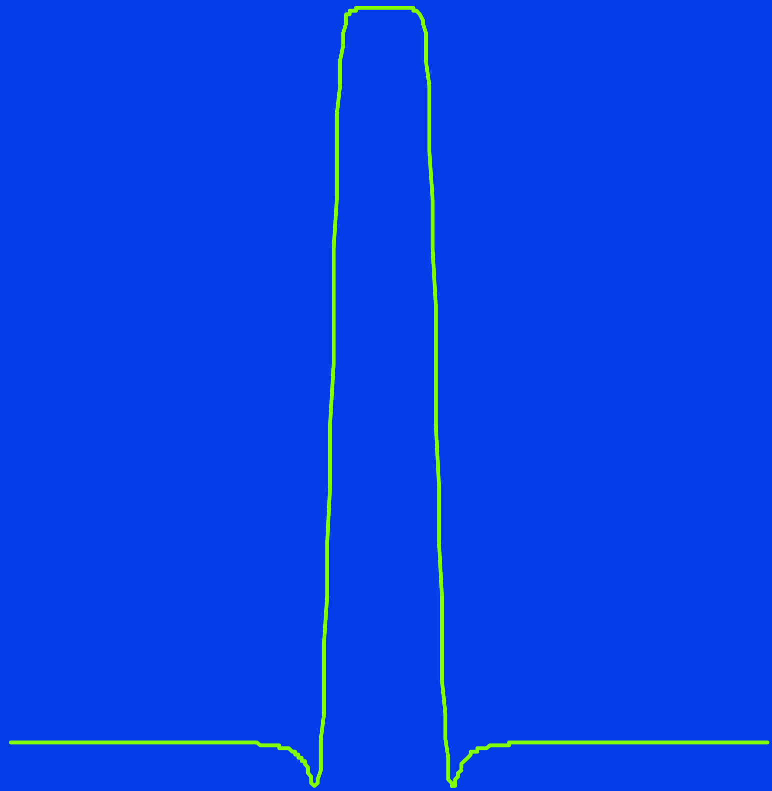
Selective (soft) Pulse

- used as a flip pulse, 30 degrees
- profile (Mxy):



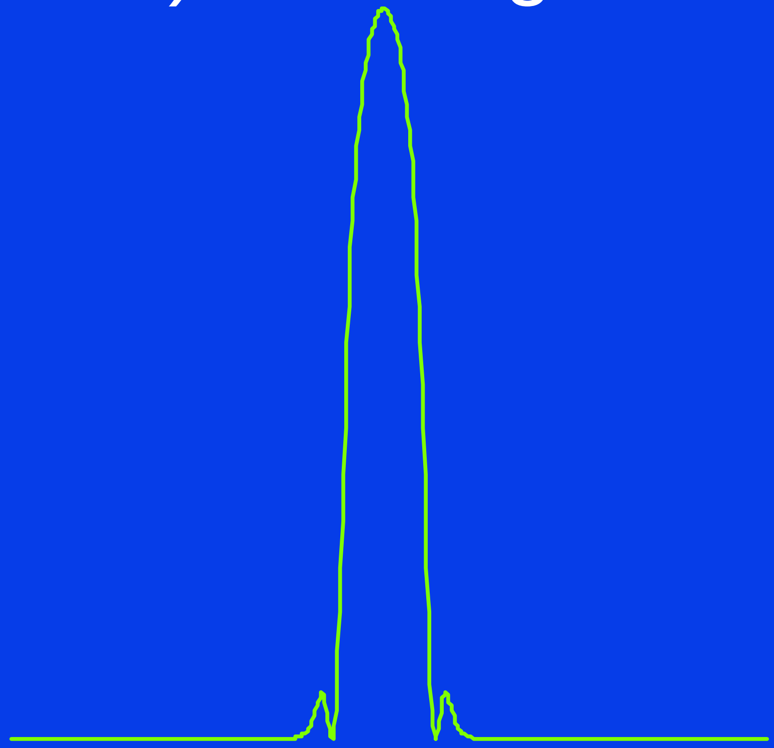
Selective (soft) Pulse

- used as a flip pulse 90 degrees:



Selective (soft) Pulse

- used as a SE refocusing pulse (pancake), 180 degress:



Lesson:

- The non-linearity of the Bloch equations makes it difficult to tell:
  - What slice profile an RF pulse produces (analysis)
  - More importantly: what RF pulse produces a given slice profile (synthesis)

Solution(s)

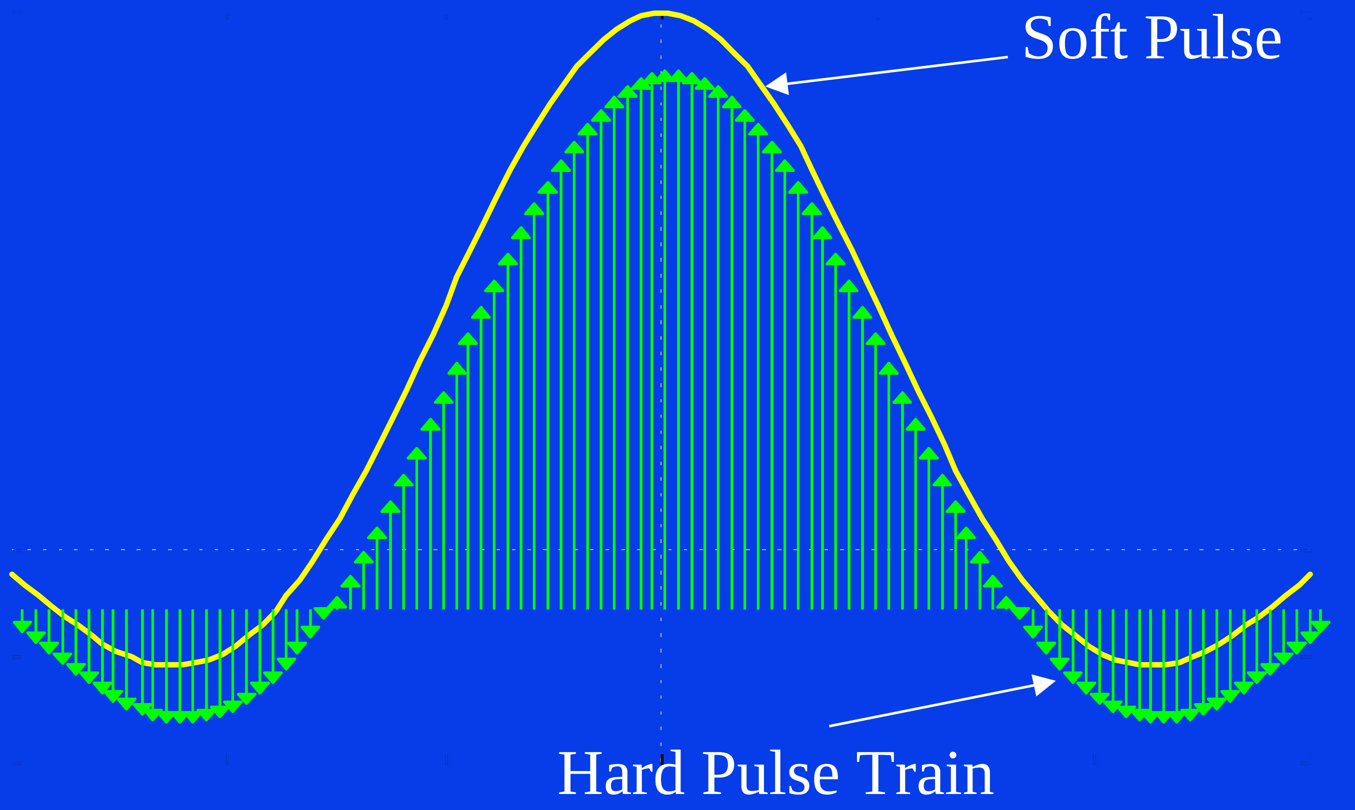
- Optimization + Bloch Equation: weeks to months J.B.Murdoch,A.H. Lent,M.R.Kritzer ‘Computer-optimized Narrowband Oulses for Multislice Imaging’ J.Mag.Res. 74,226-263 (1987)
- SLR: seconds to minutes
  - ‘Parameter Relations for the Shinnar-Le_Roux Selective Excitation Pulse Design Algorithm’,J.Pauly,P.Le_Roux,D.Nishimura,A.Macovski,IEEE Trans. Med. Imaging, vol 10, No1,March 1991

SLR, the ingredients:

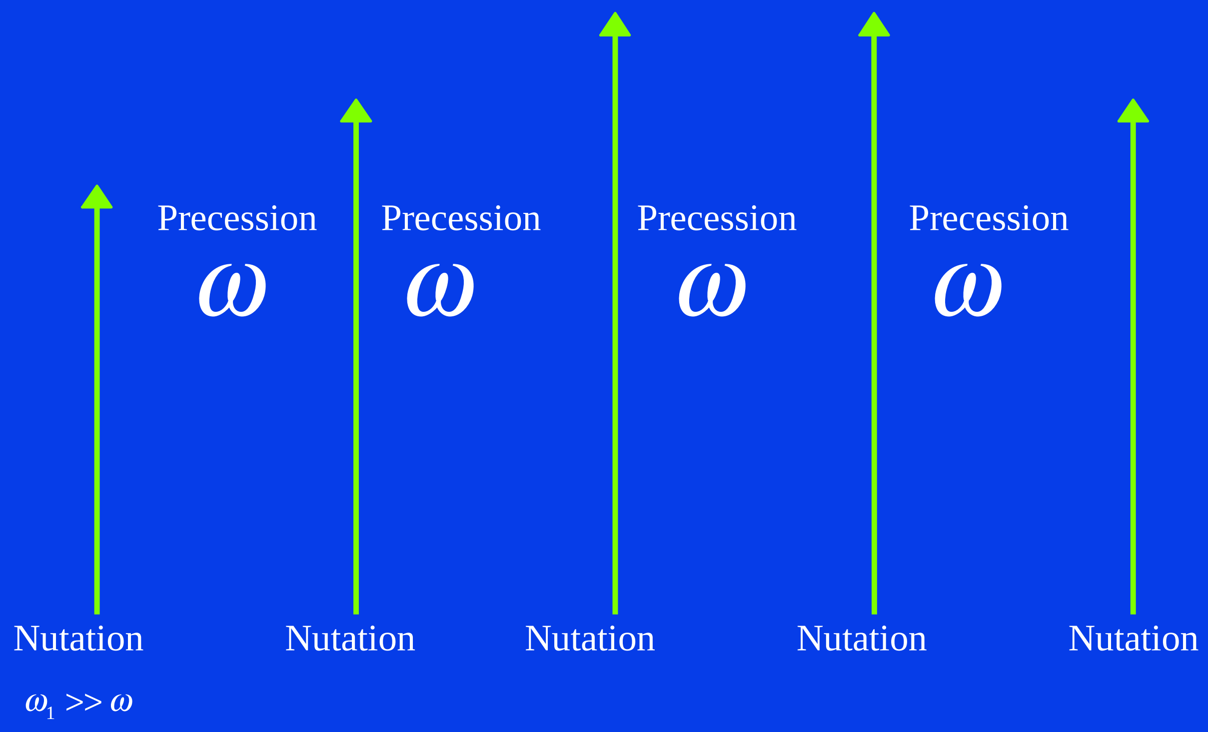
- Discrete Time Approximation : from continous to discrete time signals.

- SU2/Spinor/Quantum Mechanics/Schroedinger Equation formalism: another way to represent rotations.

Hard Pulse Train Approximation

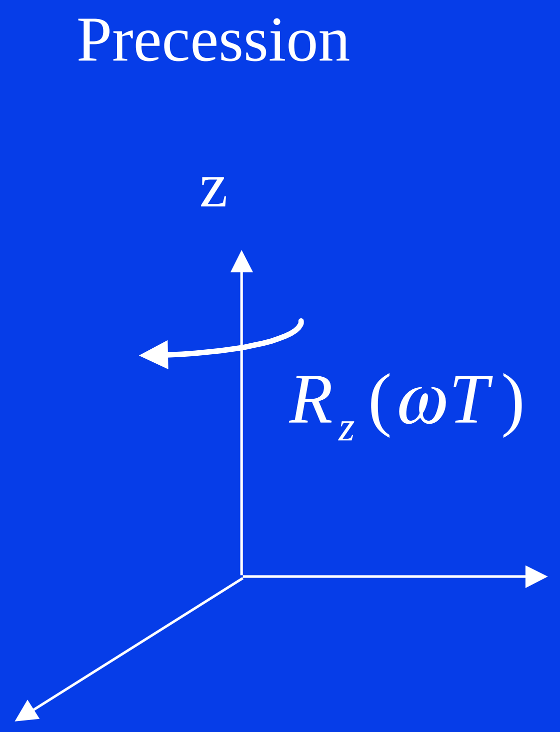


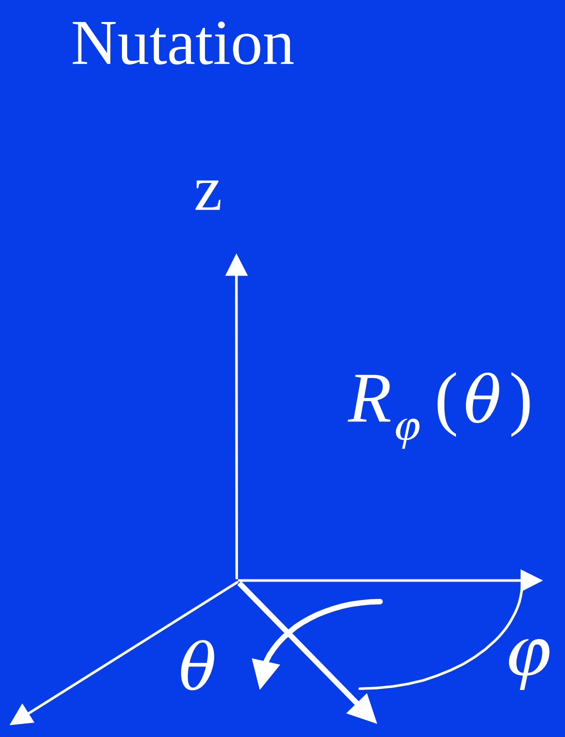
Hard Pulse Train Approximation



$$R = R_{\varphi_n}(\theta_n) \circ R_z(\omega T) \circ R_{\varphi_{n-1}}(\theta_{n-1}) \circ \cdots\cdots R_z(\omega T) \circ R_{\varphi_0}(\theta_0)$$

Hard Pulse Train Approximation





Rotations Representation

- O3 group/ Bloch equation
  - General rotation: $R = R_z(\varphi) \circ R_y(\theta) \circ R_z(\psi)$

$$\begin{bmatrix} & & \\ & O_{3x3} & \\ & & \end{bmatrix}^{+} = \begin{bmatrix} \cos\varphi\cos\theta\cos\psi - \sin\varphi\sin\psi & -\cos\varphi\cos\theta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta\cos\psi + \cos\varphi\sin\psi & -\sin\varphi\cos\theta\sin\psi + \cos\varphi\cos\psi & \sin\varphi\sin\theta \\ -\sin\theta\cos\psi & -\sin\theta\sin\psi & \cos\theta \end{bmatrix} \begin{bmatrix} & & \\ & O_{3x3} & \\ & & \end{bmatrix}$$

Rotations Representation

- SU2 / Schroedinger Equation
  - General Rotation $R = R_z(\varphi) \circ R_y(\theta) \circ R_z(\psi)$

    2x2 complex matrix; actually 2 complex (α, β) : spinor

$$S = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} = \begin{bmatrix} e^{-j\frac{\varphi+\psi}{2}} \cos\frac{\theta}{2} & -e^{-j\frac{\varphi-\psi}{2}} \sin\frac{\theta}{2} \\ e^{j\frac{\varphi-\psi}{2}} \sin\frac{\theta}{2} & e^{j\frac{\varphi+\psi}{2}} \cos\frac{\theta}{2} \end{bmatrix}$$

$$\alpha\,\alpha^* + \beta\,\beta^* = 1$$

Density matrix in Q.M.

Or from SU2 to O3:

$$M = \begin{bmatrix} M_z & M_{xy}^* \\ M_{xy} & -Mz \end{bmatrix}$$

$$\underbrace{M^+ = SMS^* = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \begin{bmatrix} M_z & M_{xy}^* \\ M_{xy} & -M_z \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix}}_{\text{Equivalent to Bloch equations}}$$

(density matrix in Quantum Mechanics)

Rotation SU2: precession

- Precession= Nutation of angle φ around Z:

$$S = \begin{bmatrix} e^{-j\varphi/2} & 0 \\ 0 & e^{j\varphi/2} \end{bmatrix}$$

$$\begin{bmatrix} M_z & -M_{xy}^* \\ M_{xy} & -M_z \end{bmatrix}^+ = \begin{bmatrix} e^{-j\varphi/2} & 0 \\ 0 & e^{j\varphi/2} \end{bmatrix} \begin{bmatrix} M_z & -M_{xy}^* \\ M_{xy} & -M_z \end{bmatrix} \begin{bmatrix} e^{j\varphi/2} & 0 \\ 0 & e^{-j\varphi/2} \end{bmatrix} = \begin{bmatrix} M_z & -(e^{j\varphi} M_{xy})^* \\ e^{j\varphi} M_{xy} & -M_z \end{bmatrix}$$

$$M_z^+ = M_z \qquad M_{xy}^+ = e^{j\varphi} M_{xy}$$

Rotations: SU2: nutation around y

- Nutation= rotation of angle θ around y

$$S = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$c = \cos(\theta/2)$$

$$s = \sin(\theta/2)$$

Density Matrix == Bloch ?

$$\begin{bmatrix} M_z^+ & * \\ M_{xy}^+ & * \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} M_z & M_{xy}^* \\ M_{xy} & -M_z \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$M_z^+ = (c^2 - s^2) M_z - cs\, M_{xy}^* - cs\, M_{xy}$$

$$M_{xy}^+ = 2cs\, M_z - s^2 M_{xy}^* + c^2 M_{xy}$$

Rotations: SU2

- Sequence of rotations still represented by matrix multiplication:

$$R = R_2 \circ R_1$$

$$M^+ = \underbrace{S_2 \underbrace{(S_1 M S_1^*)}_{\text{Bloch } R_1} S_2^*}_{\text{Bloch } R_2}$$

$$M^+ = (S_2 S_1) M (S_2 S_1)^*$$

$$R = R_2 \circ R_1 \leftrightarrow S = S_2 S_1$$

Rotations: spinors

As the SU2 matrices are completely defined by two complex numbers we can even forget the matrix notation, defining the ‘spinors’ as the two components of the first column of the SU2 matrix:

$$(\alpha, \beta)$$

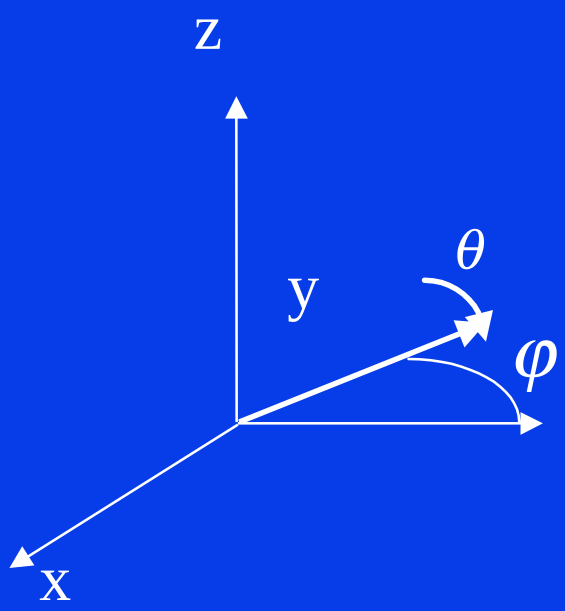
With the ‘multiplication’ operation, corresponding to the multiplication of two SU2 matrix:

$$(\alpha_2, \beta_2) * (\alpha_1, \beta_1) = (\alpha_2\alpha_1 - \beta_2^* \beta_1, \beta_2\alpha_1 + \alpha_2^* \beta_1)$$

For all purposes we will keep the matrix notation; knowing that ,when the operations have to be performed, we only need the first column of the result.

SU2 Rotations: general nutation

Nutation



$$R_{\varphi}(\theta) =$$

$$R_z(\varphi) \circ R_y(\theta) \circ R_z(-\varphi)$$

SU2 rotations: general nutation

- Nutation (2): rotation of angle θ around an axis in x,y:

$$S=\underbrace{\begin{bmatrix} e^{-j\varphi/2} & 0 \\ 0 & e^{j\varphi/2} \end{bmatrix}}_{R_z(\varphi)}\underbrace{\begin{bmatrix} c & -s \\ s & c \end{bmatrix}}_{R_y(\theta)}\underbrace{\begin{bmatrix} e^{j\varphi/2} & 0 \\ 0 & e^{-j\varphi/2} \end{bmatrix}}_{R_z(-\varphi)}=\begin{bmatrix} c & -se^{-j\varphi} \\ se^{j\varphi} & c \end{bmatrix}$$

or making s complex:

$$S=\begin{bmatrix} c & -s^{*} \\ s & c \end{bmatrix}$$

SLR Direct Recursion (Analysis)

Precession $-\omega T$

$$\begin{bmatrix} e^{j\omega T/2} & 0 \\ 0 & e^{-j\omega T/2} \end{bmatrix}$$

Precession $-\omega T$

$$\begin{bmatrix} Z^{1/2} & 0 \\ 0 & Z^{-1/2} \end{bmatrix}$$

Nutation

$$\begin{bmatrix} C_{i-1} & -S_{i-1}^{*} \\ S_{i-1} & C_{i-1} \end{bmatrix}$$

Nutation

$$\begin{bmatrix} C_{i} & -S_{i}^{*} \\ S_{i} & C_{i} \end{bmatrix}$$

Nutation

$$\begin{bmatrix} C_{i+1} & -S_{i+1}^{*} \\ S_{i+1} & C_{i+1} \end{bmatrix}$$

SLR Direct Recursion

From after pulse i-1 to after pulse i:

$$\begin{bmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix} = \begin{bmatrix} c_i & -s_i^* \\ s_i & c_i \end{bmatrix} \begin{bmatrix} Z^{1/2} & 0 \\ 0 & Z^{-1/2} \end{bmatrix} \begin{bmatrix} \alpha_{i-1} & -\beta_{i-1}^* \\ \beta_{i-1} & \alpha_{i-1}^* \end{bmatrix}$$

$$Z = e^{j\omega T}$$

$$c_i = \cos(\theta_i/2)$$

$$s_i = e^{j\varphi_i} \sin(\theta_i/2)$$

SLR Direct Recursion

$$\begin{cases} \alpha_i = c_i Z^{1/2} \alpha_{i-1} - s_i^* Z^{-1/2} \beta_{i-1} \\ \beta_i = s_i Z^{1/2} \alpha_{i-1} + c_i Z^{1/2} \beta_{i-1} \end{cases}$$

$$\alpha_i = Z^{i/2} a_i$$

$$\beta_i = Z^{i/2} b_i$$

$$\begin{cases} a_i = c_i a_{i-1} - s_i^* Z^{-1} b_{i-1} \\ b_i = s_i a_{i-1} + c_i Z^{-1} b_{i-1} \end{cases}$$

SLR Direct Recursion

After first pulse: $a_0 = c_0 \qquad b_0 = s_0$

After second pulse: $a_1 = c_1 c_0 - s_1^* s_0 Z^{-1} \quad b_1 = s_1 c_0 + c_1 s_0 Z^{-1}$

....

After pulse # i Polynomials of order i in Z^{-1}

SLR Direct Recursion (Analysis)

$$a_i(Z) = a_i^0 + a_i^1 Z^{-1} + a_i^2 Z^{-2} + \cdots\cdots + a_i^i Z^{-i}$$

$$b_i(Z) = b_i^0 + b_i^1 Z^{-1} + b_i^2 Z^{-2} + \cdots\cdots + b_i^i Z^{-i}$$

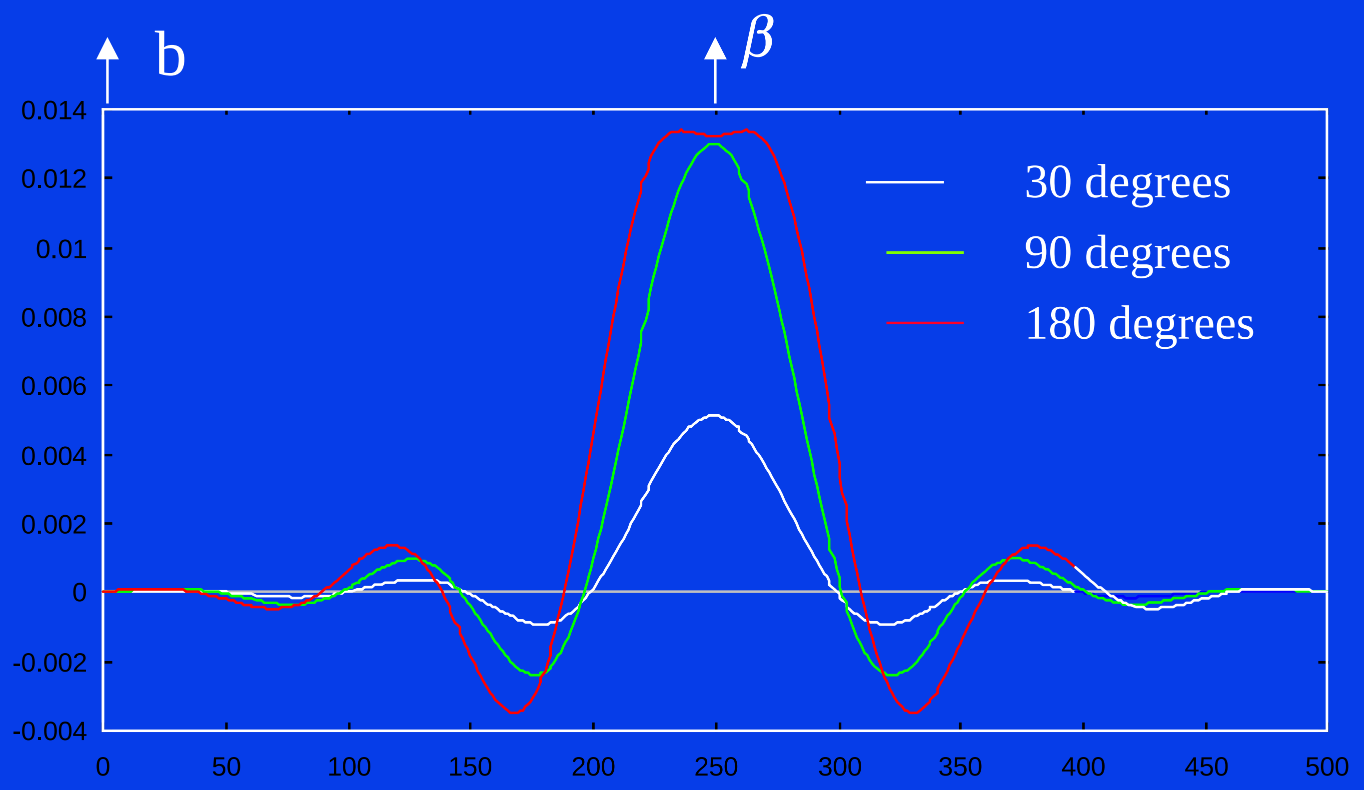
$$a_i^k = c_i a_{i-1}^k - s_i^* b_{i-1}^{k-1}$$

$$b_i^k = s_i a_{i-1}^k + c_i b_{i-1}^{k-1}$$

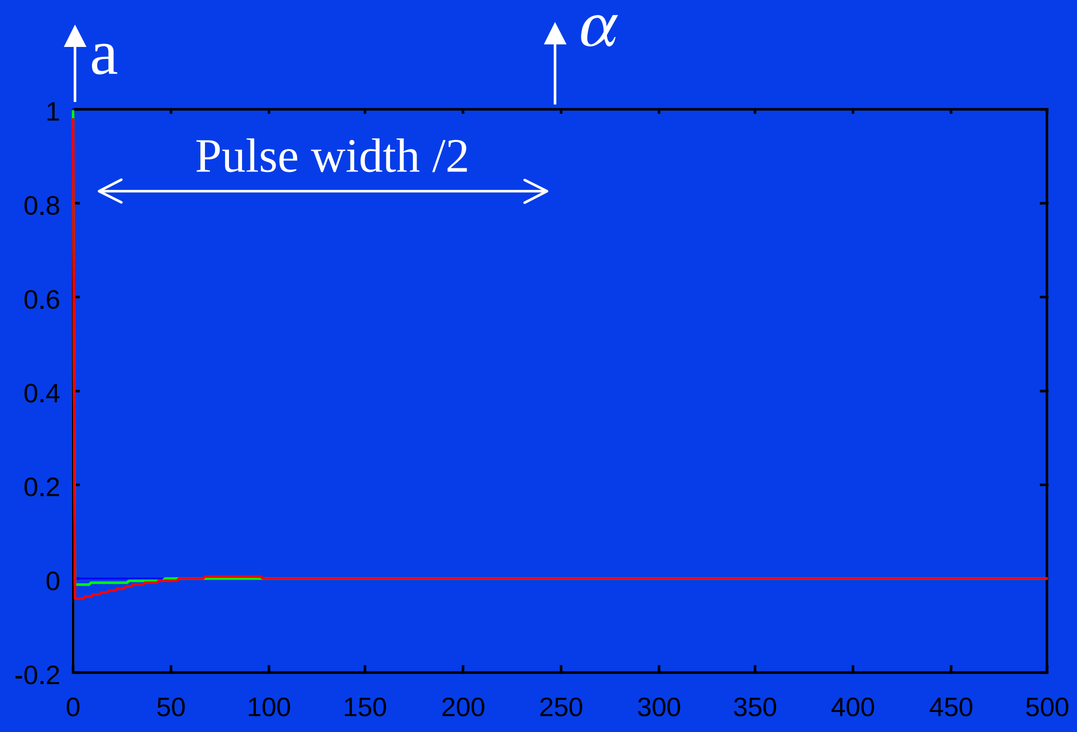
$$a_n^k, k = 0, n \xrightarrow{DFT} a(\omega)$$

$$b_n^k, k = 0, n \xrightarrow{DFT} b(\omega)$$

SLR Direct Recursion (Analysis)

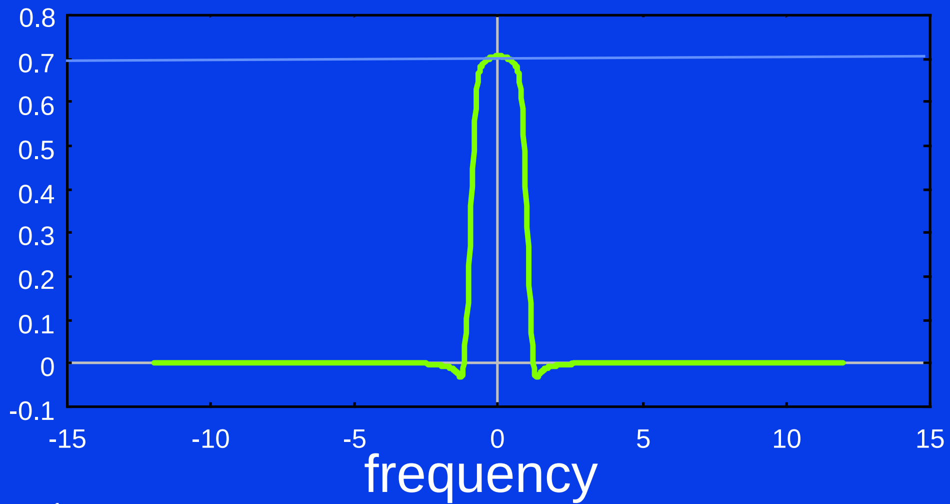


SLR Direct Recursion (Analysis)



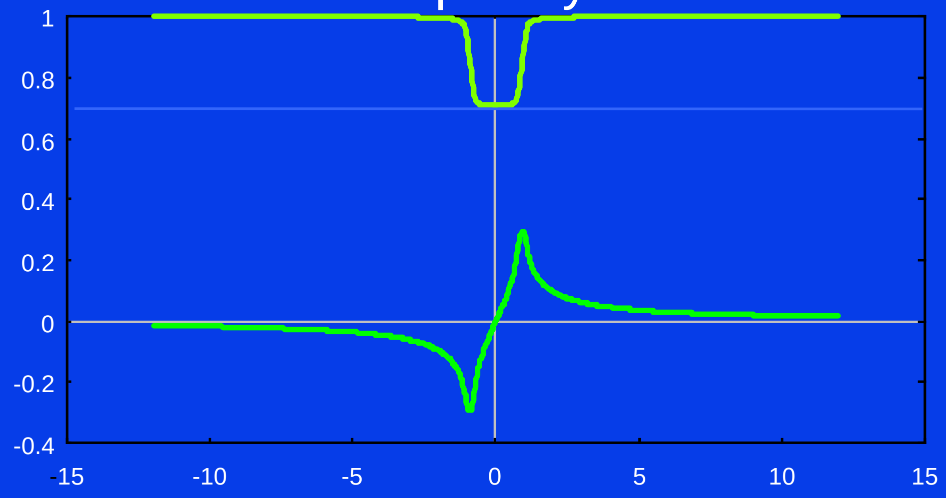
SLR Direct Recursion (Analysis)

β



$$|\beta| = \sin(\theta/2)$$

a



$$|\alpha| = \cos(\theta/2)$$

SLR Inverse Recursion

Given two polynomials of order n, a and b such that $aa^* + bb^* = 1$

Can we find the sequence of (θ_i, φ_i) (or equivalently s_i) which yields (by the SLR direct recursion) this a and b?

The answer is yes

SLR Inverse Recursion

'Matrix' inverse the direct recursion:

$$\begin{bmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix} = \begin{bmatrix} c_i & -s_i^* \\ s_i & c_i \end{bmatrix} \begin{bmatrix} Z^{1/2} & 0 \\ 0 & Z^{-1/2} \end{bmatrix} \begin{bmatrix} \alpha_{i-1} & -\beta_{i-1}^* \\ \beta_{i-1} & \alpha_{i-1}^* \end{bmatrix}$$

obtaining:

$$\begin{bmatrix} \alpha_{i-1} & -\beta_{i-1}^* \\ \beta_{i-1} & \alpha_{i-1}^* \end{bmatrix} = \begin{bmatrix} Z^{-1/2} & 0 \\ 0 & Z^{1/2} \end{bmatrix} \begin{bmatrix} c_i & s_i^* \\ -s_i & c_i \end{bmatrix} \begin{bmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix}$$

SLR Inverse Recursion

Considering only the first column (spinor) and in terms of a and b

$$a_{i-1} = c_i a_i + s_i^* b_i$$

$$b_{i-1} = Z^{-1}(-s_i a_i + c_i b_i)$$

Order -1 of b_{i-1} and order i of a_{i-1} should be zeroed !

SLR Inverse Recursion

$$t_i = \frac{s_i}{c_i} = \frac{b_i^0}{a_i^0} \qquad t_i^* = \frac{s_i^*}{c_i} = -\frac{a_i^i}{b_i^i}$$

2 equations , 1 unknown ! Yes but:

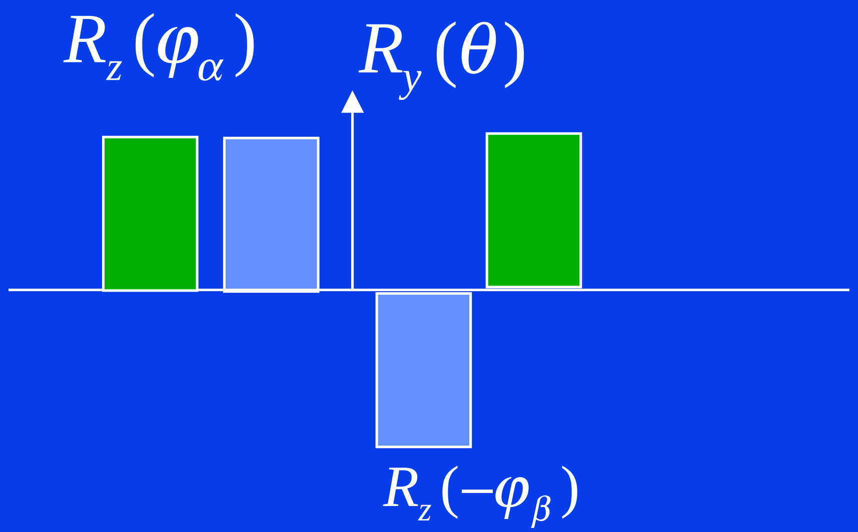
$$a_i a_i^* + b_i b_i^* = 1 \quad \Rightarrow \quad a_i^{0*} a_i^i + b_i^{0*} b_i^i = 0$$

1 equation, 1 unknown.

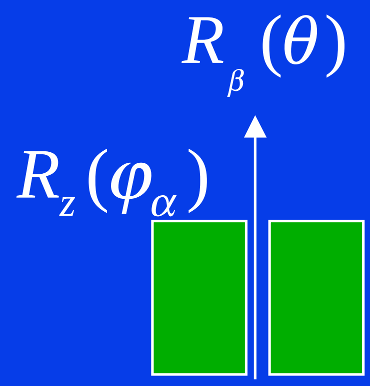
Synthesis: filters (polynomials) specifications

$$S(\omega)=\begin{bmatrix}\alpha & -\beta^* \\ \beta & \alpha^*\end{bmatrix}=\begin{bmatrix}e^{j\varphi_\alpha}\cos(\theta/2) & -e^{-j\varphi_\beta}\sin(\theta/2) \\ e^{j\varphi_\beta}\sin(\theta/2) & e^{-j\varphi_\alpha}\cos(\theta/2)\end{bmatrix}$$

$$\boxed{\sin(\theta(\omega)/2)=|\beta|}$$



OR:



From Spinor <-> Magnetization

- Flip pulse/Inversion/Saturation: Initial Magnetization on Z. Using density matrix equation:

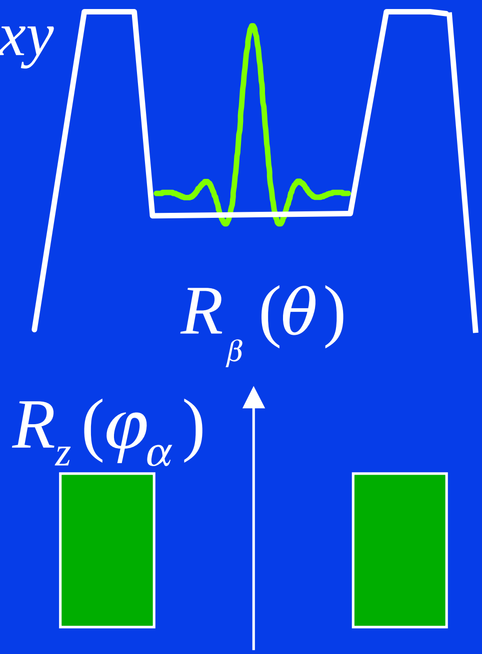
$$M_{xy}^{+} = 2\alpha^{*}\beta = 2|\beta|\sqrt{1-|\beta|^{2}}\,e^{j\varphi_\beta}e^{-j\varphi_\alpha}$$

$$M_{z}^{+} = \alpha^{*}\alpha - \beta^{*}\beta = 1-2|\beta|^{2}$$

From Spinor to Magnetization

- Refocusing pulse: initial magnetization in xy

$$M_{xy} = -\beta^2 M_{xy}^* + \alpha^{*2} M_{xy}$$



Crush:

$$M_{xy} = -\beta^2 M_{xy}^*$$

Synthesis, 1st step

- What $\beta(\omega), \alpha(\omega)$ should we take as target?

$\beta(\omega)$ is easy to specify, according to type of pulses by $|\beta| = \sin(\theta/2)$

Outside band: $\beta(\omega)$ =0

Synthesis 1st step

In Band:

	flip	saturat	Invers.	Refoc.
$\lvert\beta\rvert$	$\sin(\theta/2)$	$1/\sqrt{2}$	1	1
$\angle\beta$	0	free	free	0

Synthesis

b must be polynomial of order n -> FIR filter design algorithm: whether by windowing (a la sinc) or by a more optimal algorithm (Remez,Parks-McClellan)

What about a?

if b is chosen, then the magnitude of a is given

$$aa^*(\omega) = 1 - bb^*(\omega)$$

But, what phase ?

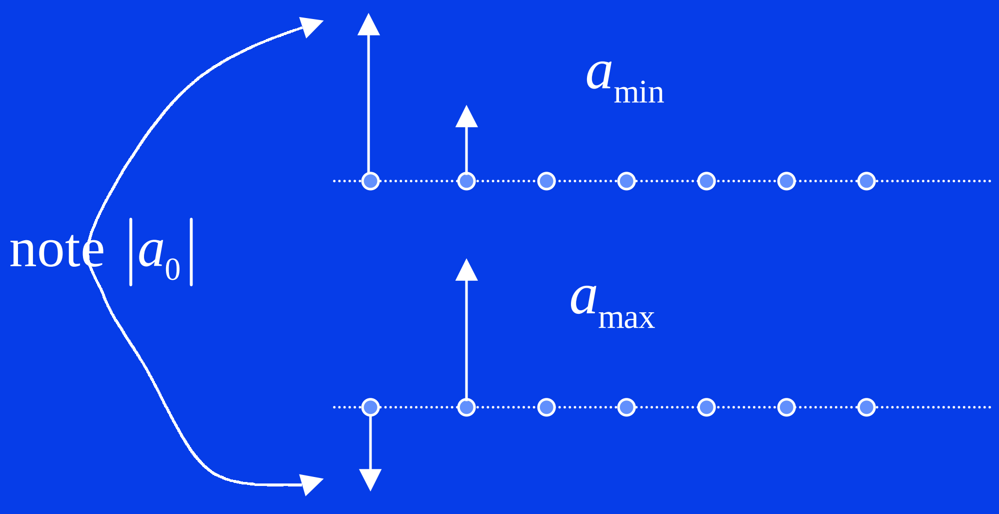
Minimum phase solution <==> Pulse with minimum energy

polynomials of order 1

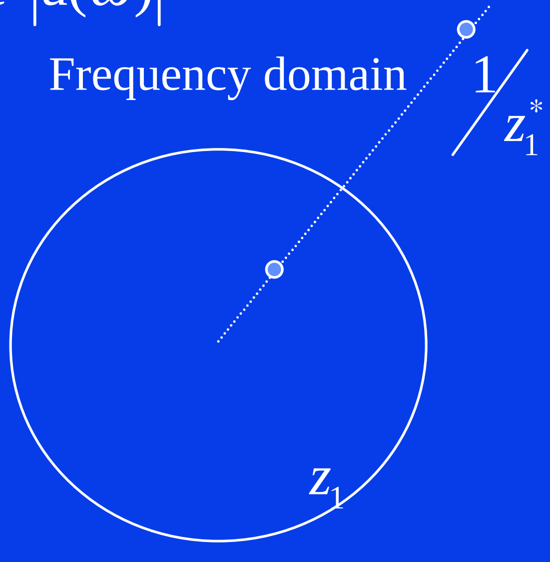
$$a_{\min} = 1 - z_1 Z^{-1}$$

$$a_{\max} = -z_1^* + Z^{-1} \qquad (= Z^{-1} a_{\min}{}^*) \rightarrow \text{same } |a(\omega)|^2$$

‘Time’ domain



Frequency domain



Synthesis 2nd step: a

$$a = k(1 - z_1 Z^{-1}) \quad (1 - z_2 Z^{-1}) \cdots \quad (1 - z_n Z^{-1})$$

$$\text{or } (-z_1^* + Z^{-1}) \text{ or } (-z_2^* + Z^{-1}) \cdots \text{or } (-z_n^* + Z^{-1})$$

Min phase choice: all zeroes inside the unit circle -> 1) Frequency property: we do not have to know the zeroes to find its phase:

$$\varphi_a(\omega) = \mathrm{H}(\log|a|(\omega))$$

See the companion technote or Oppenheim Schaeffer 'Digital Signal Processing'

2) Time domain property: the leading term a0 is MAXIMUM.

From the SLR direct recusrion: $a_0 = c_0 c_1 \ldots c_n$

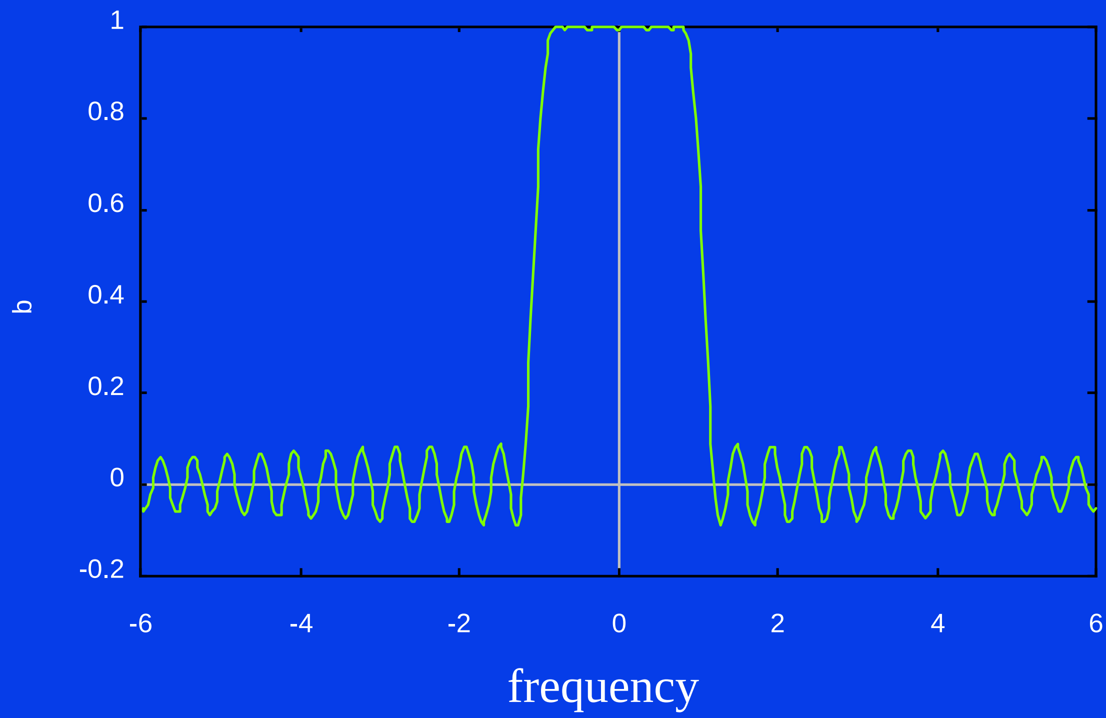
$$\log a_0 = \sum_{i=0}^{n} \log(\cos(\frac{\theta_i}{2})) \simeq -\frac{1}{8} \sum_{i=0}^{n} \theta_i^2$$ **Min Energy**

Synthesis last step

Run the SLR inverse recursion: 10 lines of code.

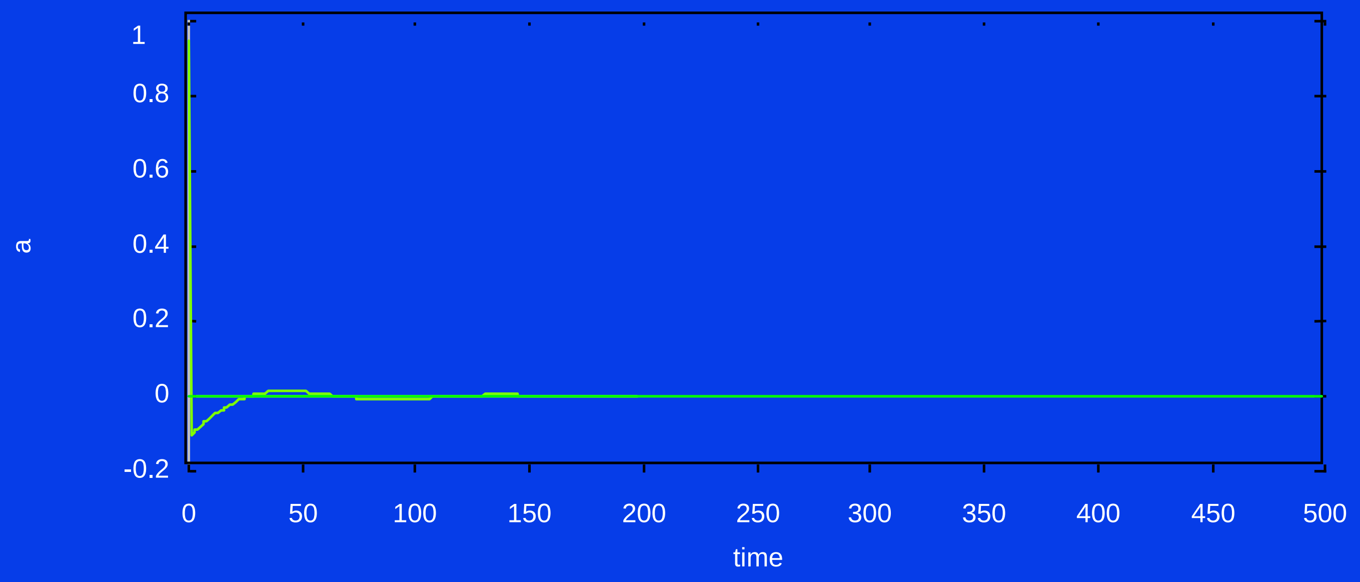
Example: Pancake Pulse

- Design b, by Remez algorithm



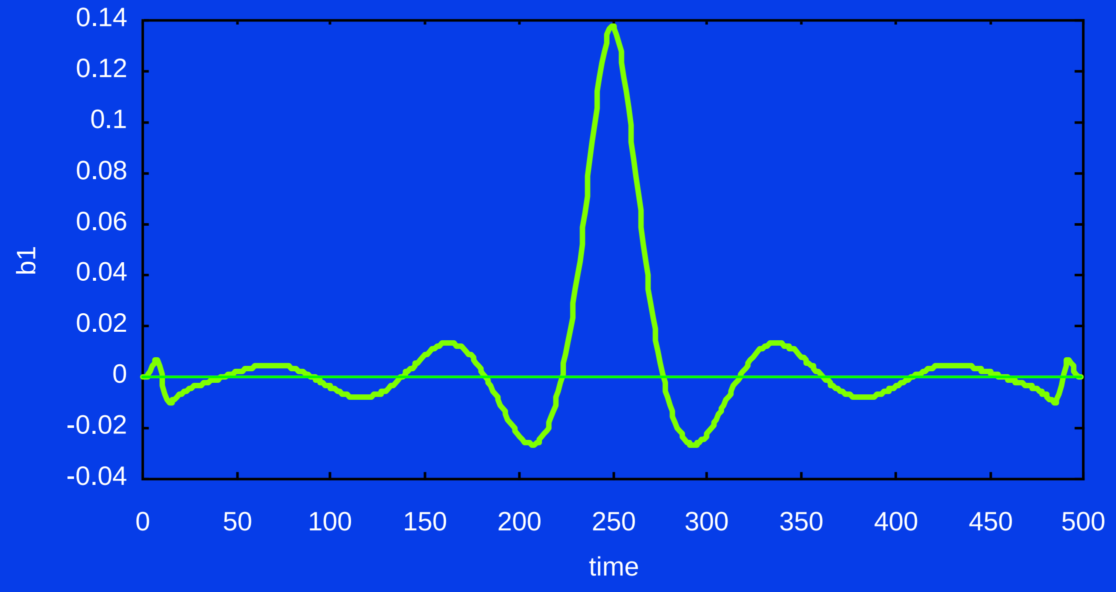
Example: Pancake Pulse

- Calculate signal a (min-phase)



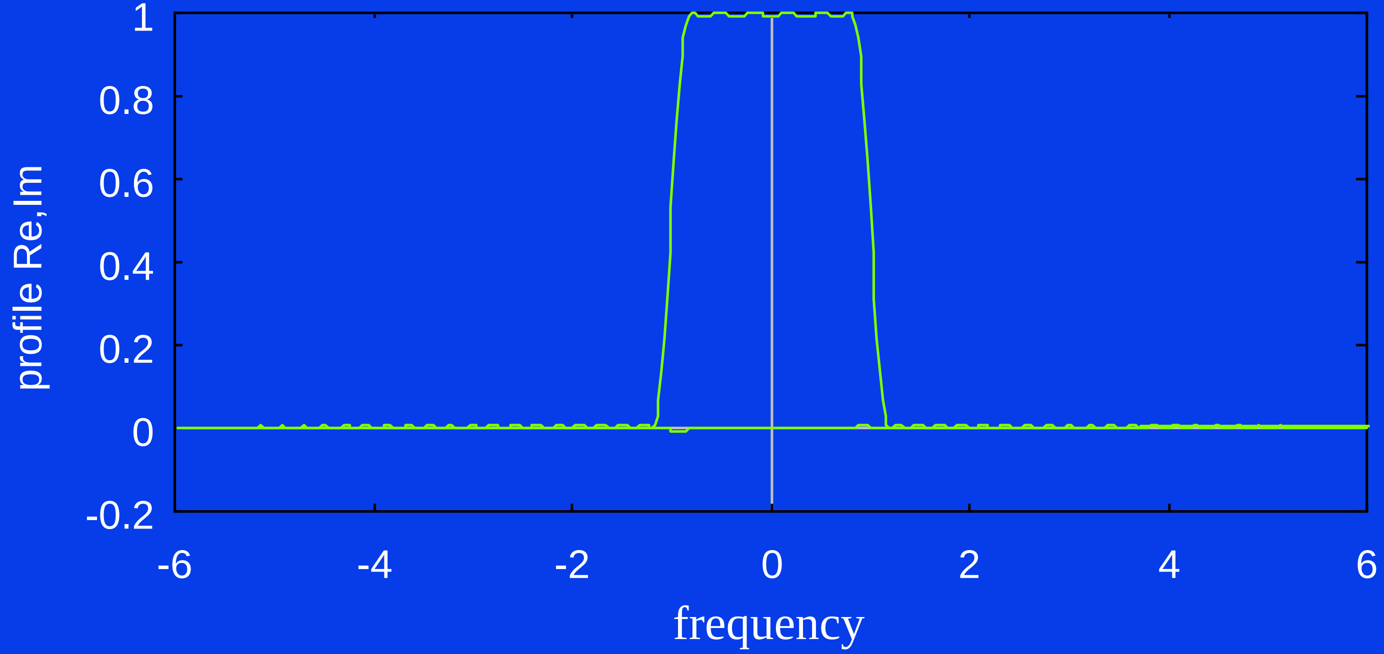
Example: Pancake Pulse

- Run Inverse SLR, obtain b1



Example: Pancake Pulse

- Verify Profile, possibly by running direct SLR and taking $-\beta^2$



Subsequent Developments

- Phase massaging by reflecting zeroes, S.Pickup,X.Ding MRM 33:648-655, 1995.
- Self refocusing pulses (choice of zeroes of a such that it cancel the phase of b, in the pass-band): J.M. Pauly and A. Macovski.Proc. Tenth SMRM, page 267, 1991.
- FIR filter design for quadratic phased β ,P.LeRoux et al JMRI 1998;8:1022-1032.

Challenges

1) Adiabatic (B1 insensitive) pulses

2) Take whether a short train of high energy pulses (around 180 degrees) or a very long train of small pulse, in short a very high energy pulse train: a and b have, in time domain, all their energy centered around the middle: the SLR inversion is numerically unstable. SLR relies on a being min phase or close to min phase !!!

THANK YOU FOR YOUR ATTENTION !

A pdf version of this presentation and some other information is available at:

http://plenag.free.fr

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